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April 1982

# EXACT SOLUTION FOR THE ELECTROMAGNETIC FIELDS OF A UNIFORM LINE-CURRENT PARALLEL TO A FLAT HOMOGENEOUS EARTH

Edward A. Lewis
John L. Heckscher

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A detailed step-by-step description is given of the exact solution of the boundary-value problem of calculating the fields of a uniform, sinusoidally varying, straight-line current flowing at a fixed height above a flat homogeneous earth of arbitrary dielectric constant and conductivity. The solution satisfies Maxwell's equations and boundary conditions, reduces to Ampere's Law at the current itself, and hence includes the radiation fields. The results contain integrals with infinite limits, but these can be evaluated numerically

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20. Abstract (Continued)

in specific cases of interest. As an example, the resistivity induced in a wire 10 m above the earth of $10^{-3}$ mho/m conductivity is calculated for a frequency of 35 kHz.

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# Contents



# Illustrations

# Exact Solution For The Electromagnetic Fields of a Uniform Line Current Parallel to a Flat Homogeneous Earth

## 1. INTROD CTION

In his pioneer paper, Carson[1] obtained approximate solutions for essentially the same problem addressed here, but the magnitude of the errors caused by simplifying assumptions, such as neglecting displacement currents, is unclear, especially in cases of poor conductivity and high frequencies. The alternate approach adopted here is to develop the exact solutions which, as might be expected, are more cumbersome. They do, however, offer the possibility of numerical evaluation to a desired degree of accuracy in specific cases.

## 2. APPROACH

Figure 1 represents a rectangular coordinate system XYZ, with the XOZ plane coincident with the flat earth-surface. A line-current of I rms amperes flows in the positive Z-direction at height h, and passes through the point $x = 0$, $y = h$, $z = 0$. The material of the uniform earth has a dielectric constant $\varepsilon$ F/m and a conductivity of $\sigma$ mho/m. The region $y > 0$ is taken to be free space of

(Received for publication 5 April 1982)

1. Carson, J.R. (1926) Wave propagation in overhead wires with ground return, Bell System Technical Journal 5:539-554.

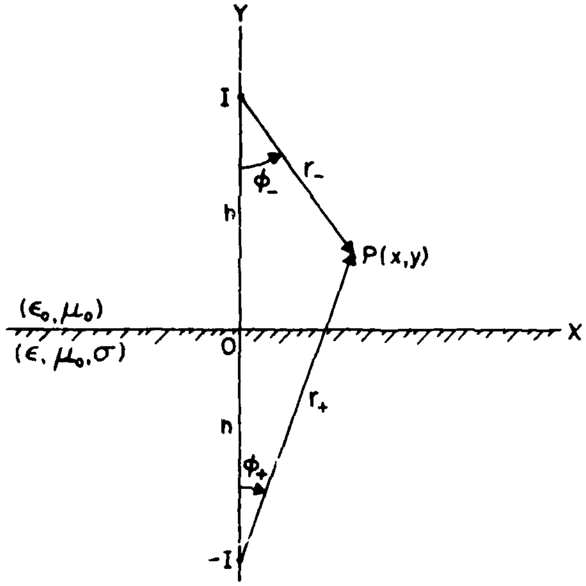


Figure 1. Coordinates in a Plane Transverse to the Z-axis, Which is Directed Toward the Reader

dielectric constant $\varepsilon_0$, and both media have the magnetic permeability $\mu_0$ of free space. MKS units are used throughout.

The field at an arbitrary point P(x, y, z) above the earth is regarded as consisting of three parts: (1) the fields which the current would produce if it were in free space, the earth being absent; (2) the fields of a perfect image of the current at $x = 0$, $y = -h$, as if the earth were perfectly conducting; and (3) "supplementary" fields which account for the earth not being a perfect conductor. The fields inside the earth have no source-singularity and are described as just one field-complex. The free-space field is discussed in Section 3, the effects of the image are obtained in Section 4, and the supplementary fields are developed in Section 5.

## 3. FREE SPACE FIELDS

In discussing the free-space fields it is convenient to use an auxiliary cylindrical coordinate system $(r_-, \phi, z)$ whose z-axis coincides with the line-current, as shown in Figure 1. The current I and all its fields vary with time t according to the factor $\epsilon^{-i\omega t}$ (which is not explicitly written out in the expressions to follow)

where $\omega$ is the angular frequency corresponding to the current frequency $f$. In free space, Maxwell's equations for the electric field $\vec{E}$ and the magnetic field $\vec{H}$ are

$$\nabla \times \vec{E} = i\omega\mu_o \vec{H} \tag{1}$$

$$\nabla \times \vec{H} = -i\omega\varepsilon_o \vec{E} \tag{2}$$

Because of symmetry and the absence of free charges, only the field components $E_z$ and $H_\phi$ exist, and these do not vary with z. In cylindrical coordinates the Maxwell equations become

$$\frac{\partial E_z}{\partial r_-} = -i\omega\mu_o H_\phi \tag{3}$$

$$\frac{1}{r_-}\frac{\partial}{\partial r_-}(r_- H_\phi) = -i\omega\varepsilon_o E_z \tag{4}$$

The Hankel functions $H_o^{(1)}$ and $H_1^{(1)}$ are related by the recurrence relations[2]

$$\frac{dH_o^{(1)}(\upsilon)}{d\upsilon} = -H_1^{(1)}(\upsilon) \tag{5}$$

$$\frac{1}{\upsilon}\frac{d}{d\upsilon}\left\{\upsilon H_1^{(1)}(\upsilon)\right\} = H_o^{(1)}(\upsilon) \tag{6}$$

Using these it is readily shown that Eqs. (3) and (4) are satisfied by

$$E_z = -\frac{\mu_o\omega I}{4} H_o^{(1)}(kr_-) \tag{7}$$

$$H_\phi = \frac{ikI}{4} H_1^{(1)}(kr_-) \tag{8}$$

where

$$k = \omega\sqrt{\varepsilon_o\mu_o} = \frac{\omega}{c} = \frac{2\pi}{\lambda} \tag{9}$$

2. Abramowitz, A., and Stegun, I.A. (1964) Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55, U.S. Government Printing Office, Washington, D.C.

where c is the velocity of electromagnetic waves in free space, and $\lambda$ is the wavelength.

Other properties of the Hankel and Bessel functions of the first kind (J), and the second kind[†] (Y) which will be needed are:

$$H_o^{(1)} = J_o + i\,Y_o$$

$$H_1^{(1)} = J_1 + i\,Y_1$$

and for $\upsilon \to 0$,

$$H_o^{(1)}(\upsilon) \to 1 - \frac{2i}{\pi} \ln \frac{2}{\gamma\upsilon} \tag{10}$$

$$H_1^{(1)}(\upsilon) \to \frac{\upsilon}{2} - \frac{2i}{\pi\upsilon} + \frac{i\upsilon}{\pi} \ln \frac{\upsilon}{2}$$

where $\gamma = 1.78107$.
Also, for $\upsilon \to \infty$,

$$H_o^{(1)}(\upsilon) \to \sqrt{\frac{2}{\pi\upsilon}}\; e^{i(\upsilon - \pi/4)}$$

$$H_1^{(1)}(\upsilon) \to \sqrt{\frac{2}{\pi\upsilon}}\; e^{i(\upsilon - 3\pi/4)}$$

Close to the line current, $kr_- \to 0$ and

$$E_z \to -\frac{\mu_o \omega I}{4} \left\{ 1 - \frac{2i}{\pi} \ln \frac{2}{\gamma k r_-} \right\} \tag{11}$$

$$H_\phi \to \frac{I}{2\pi r_-} \tag{12}$$

The constants in Eqs. (7) and (8) were pre-chosen so that Eq. (12) agrees with Ampere's Law.

[†]Some writers represent this function by N.

At large distances from the line current, $kr_- \to \infty$ and

$$E_z \to -\frac{\mu_o \omega I}{\sqrt{8\pi k r_-}} e^{i(kr_- - \pi/4)} \tag{13}$$

$$H_\phi \to I\sqrt{\frac{k}{8\pi r_-}}\, e^{i(kr_- - \pi/4)} \tag{14}$$

Since the time factor is $e^{-i\omega t}$ and k is a positive real number these fields have the form of outgoing waves, as they should. Thus the fields given by Eqs. (7) and (8) meet all the requirements for an exact solution for the free-space case. Also, for $kr_- \to \infty$,

$$\frac{E_z}{H_\phi} \to -\sqrt{\frac{\mu_o}{\epsilon_o}} \tag{15}$$

which is the impedance relation for plane waves in free-space. The (radiated) power flowing through a 1 m section of a large cylinder is then

$$2\pi r_- |E_z| \cdot |H_\phi| = \frac{\mu_o \omega I^2}{4} \quad . \tag{16}$$

This power loss is as if the line-current encountered a resistivity $\mu_o\omega/4$ $\Omega$/m, and it may be seen from Eq. (11) that the in-phase, or real part of $E_z$ at the line current is negative and opposes the current flow. The power required to drive the current against this field is $\mu_o\omega I^2/4$ W/m and is equal to the radiated power. The imaginary part of $E_z$ has a logarithmic singularity at $r_- = 0$, as can be seen from Eq. (11). This is a consequence of having a current of infinitely small cross section; if the current were distributed over a finite cross section the field would remain finite.

## 4. FIELDS OVER A PERFECTLY CONDUCTING EARTH

The fields inside a perfectly conducting earth are of course zero, as is the tangential component of the electric field at the surface. The boundary conditions are met by placing a 180 degree out-of-phase image at x = 0, y = -h. The fields at a point P above the surface are then a superposition of fields of the type discussed in Section 3. Thus

$$E_{z,\infty} = -\frac{\mu_o \omega I}{4}\left[H_o^{(1)}(kr_-) - H_o^{(1)}(kr_+)\right] \tag{17}$$

where the subscript $\infty$ is used to denote the case of the perfectly conducting earth, and

$$r_\pm = \sqrt{x^2 + (y \pm h)^2} \tag{18}$$

Since the magnetic fields are in the $\phi_-$ and $\phi_+$ directions, as shown in Figure 1 and since

$$\cos\phi_- = \frac{h-y}{r_-}, \qquad \sin\phi_- = \frac{x}{r_-},$$

and (19)

$$\cos\phi_+ = \frac{h+y}{r_+}, \qquad \sin\phi_+ = \frac{x}{r_+},$$

resolving the magnetic fields into x- and y-components gives

$$H_{x,\infty} = \frac{ikI}{4}\left[\frac{h-y}{r_-} H_1^{(1)}(kr_-) + \frac{h+y}{r_+} H_1^{(1)}(kr_+)\right] \tag{20}$$

$$H_{y,\infty} = \frac{ikI}{4}\left[\frac{x}{r_-} H_1^{(1)}(kr_-) - \frac{x}{r_+} H_1^{(1)}(kr_+)\right] \tag{21}$$

Just above the earth, as $y \to 0$, $E_{z,\infty}$ and $H_{y,\infty}$ vanish while

$$H_{x,\infty} = \frac{ikhI}{4} \frac{H_1^{(1)}\left(k\sqrt{x^2+h^2}\right)}{\sqrt{x^2+h^2}}, \qquad y = 0. \tag{22}$$

From Eq. (17) the electric field at the current itself ($x = 0$, $y = h$) is

$$E_{z,\infty}(0,h) = -\frac{\mu_o \omega I}{4}\left[H_o^{(1)}(0) - H_o^{(1)}(2kh)\right] \tag{23}$$

Since the real part of the Hankel function $H_o^{(1)}$ is $J_o$ and since $J_o(0) = 1$, the component of field opposing the current flow is

$$-\mathrm{Re}[E_{z,\infty}] = \frac{\mu_o \omega I}{4} [1 - J_o(2kh)] \qquad (24)$$

The power required to drive the current is I times this, and the effective resistivity is $(\mu_o\omega/4)[1 - J_o(2kh)]$ $\Omega$/m. A portion of this function is illustrated in Figure 2, which was plotted from tabulated values of $J_o$ and from the appropriate series. The physical reason for the oscillatory behavior becomes clear from considering the fields at a great distance from the current and its image. Then

$$r_{\pm} = \sqrt{x^2 + y^2 + h^2 \pm 2hy} \cong \sqrt{r^2 \pm 2hy} = r \pm h \sin\alpha \qquad (25)$$

where $r = \sqrt{x^2 + y^2}$ and $\alpha = \sin^{-1}(y/r)$ is the elevation angle of the point P. On using Eq. (13), Eq. (17) becomes

$$E_{z,\infty} \rightarrow -\frac{\mu_o \omega I}{\sqrt{8\pi k r}} e^{i(kr - \pi/4)} \left[e^{-ikh \sin\alpha} - e^{ikh \sin\alpha}\right]$$

$$= \frac{i\mu_o \omega I}{\sqrt{2\pi kr}} e^{i(kr - \pi/4)} \sin(kh \sin\alpha) \qquad (26)$$

Similarly

$$H_{\phi,\infty} \rightarrow iI\sqrt{\frac{k}{2\pi r}}\, e^{i(kr - \pi/4)} \sin(kh \sin\alpha) \qquad (27)$$

The outward power flow is

$$|E_{z,\infty}| \cdot |H_{\phi,\infty}| = \frac{\mu_o \omega I^2}{2\pi r} \sin^2(kh \sin\alpha) \;\; \mathrm{W\text{-}m^{-2}} \qquad (28)$$

Thus the current and its image produce a far-field interference pattern that changes with kh, but always has a null along the earth's surface. The power flowing through a 1 m length of a large cylinder centered on the Z-axis of the XYZ coordinate system is

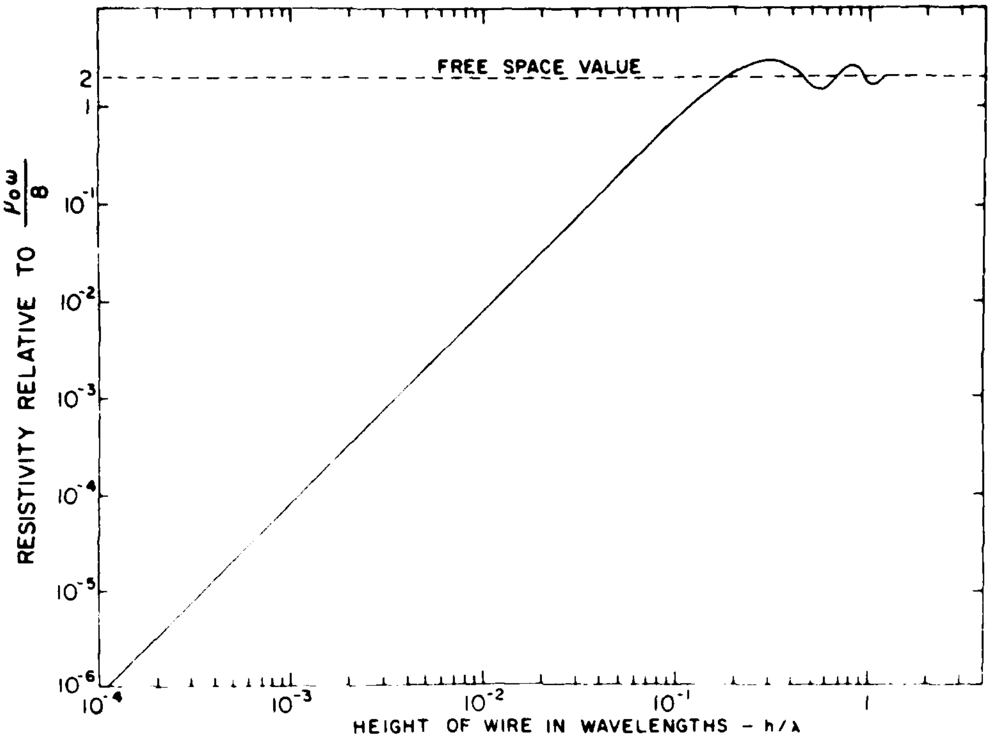


Figure 2. Radiation Resistivity of a Wire Carrying a Uniform Current as a Function of Height Above a Perfectly Conducting Flat Earth

$$\frac{\mu_o \omega I^2}{2\pi} \int_0^{\pi} \sin^2 (kh \sin \alpha)\, d\alpha = \frac{\mu_o \omega I^2}{2\pi} \int_0^{\pi/2} [1 - \cos (2kh \sin \alpha)]\, d\alpha$$

$$= \frac{\mu_o \omega I^2}{4} [1 - J_o(2kh)] \tag{29}$$

where use is made of a standard integral form. This is, of course, the same as the power expended in driving the current, as discussed above.

## 5. SUPPLEMENTARY FIELDS AND FORMAL SOLUTION

Inside the earth the Maxwell-Ampere equation is

$$\nabla \times \vec{H} = (\sigma - i\omega\varepsilon)\vec{E} \tag{30}$$

instead of Eq. (2), while Eq. (1) remains unchanged. Symmetry considerations show that the only rectangular components of the fields are $E_z$, $H_x$, and $H_y$, and these do not change with z. The Maxwell equations then reduce to

$$H_x = -\frac{i}{\omega\mu_o}\frac{\partial E_z}{\partial y} \tag{31}$$

$$H_y = \frac{i}{\omega\mu_o}\frac{\partial E_z}{\partial x} \tag{32}$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = (\sigma - i\omega\varepsilon)\, E_z \tag{33}$$

It may be shown by direct substitution that these equations are satisfied by the following expressions:

$$E_z(x, y) = \int_o^{\infty} G(s)\, e^{y\sqrt{s^2 - k_g^2}} \cos xs\, ds \tag{34}$$

$$H_x(x, y) = -\frac{i}{\omega\mu_o}\int_o^{\infty} \sqrt{s^2 - k_g^2}\; G(s)\, e^{y\sqrt{s^2 - k_g^2}} \cos xs\, ds \tag{35}$$

$$H_y(x, y) = -\frac{i}{\omega\mu_o}\int_o^{\infty} s\, G(s)\, e^{y\sqrt{s^2 - k_g^2}} \sin xs\, ds \tag{36}$$

where

$$k_g^2 = \omega^2\varepsilon\mu_o + i\sigma\omega\mu_o \tag{37}$$

and G(s) is an arbitrary function of the variable of integration, to be chosen later to satisfy the boundary conditions. In accordance with symmetry requirements $E_z$ and $H_x$ are even functions of x, while $H_y$ is an odd function. If $\sigma > 0$ the quantity $s^2 - k_g^2$ is always in the lower half of the complex plane, and hence $\sqrt{s^2 - k_g^2}$ is either in the second or fourth quadrant depending on the choice of sign. For Eqs. (34), (35), and (36) to represent physically realistic fields, the real part of $\sqrt{s^2 - k_g^2}$ must be positive so that the fields vanish as $y \to -\infty$. Thus $\sqrt{s^2 - k_g^2}$ is taken to be in the fourth quadrant.

At points above the ground, Eqs. (31), (32), and (33) with $\varepsilon = \varepsilon_o$ and $\sigma = 0$ are satisfied by the fields

$$E_z(x, y) = \int_0^\infty A(s)\, e^{-y\sqrt{s^2 - k^2}} \cos xs\, ds + E_{z,\infty} \tag{38}$$

$$H_x(x, y) = \frac{i}{\omega\mu_o} \int_0^\infty \sqrt{s^2 - k^2}\, A(s)\, e^{-y\sqrt{s^2 - k^2}} \cos xs\, ds + H_{x,\infty} \tag{39}$$

$$H_y(x, y) = -\frac{i}{\omega\mu_o} \int_0^\infty s\, A(s)\, e^{-y\sqrt{s^2 - k^2}} \sin xs\, ds + H_{y,\infty} \tag{40}$$

where $E_{z,\infty}$, $H_{x,\infty}$, and $H_{y,\infty}$ are the fields for the case of perfect ground conductivity already discussed in Section 4. The parts of the fields containing the arbitrary function A(s) individually satisfy the Maxwell equations if $k = \omega\sqrt{\varepsilon_o\mu_o}$ as before. When $s^2 > k^2$, the positive real value for $\sqrt{s^2 - k^2}$ must be chosen so that the fields decrease with increasing y. If $s^2 < k^2$, the negative value for $\sqrt{s^2 - k^2}$ must be chosen to have the solution correspond to outgoing waves rather than physically unrealistic incoming waves. As a reminder of this sign choice, an asterisk (*) is added to the square root sign in subsequent expressions.

At the earth's surface (y = 0) the tangential components of the electric and magnetic fields must be continuous. Continuity of $E_z$ requires

$$E_z(x, 0) = \int_0^\infty A(s) \cos xs\, ds = \int_0^\infty G(s) \cos xs\, ds \tag{41}$$

since $E_{z,\infty}$ vanishes at y = 0. Evidently A and G are essentially *Fourier* transforms of the same even function $E_z(x, 0)$, so that

$$A(s) = G(s) \tag{42}$$

Similarly, continuity of $H_x$ requires

$$\frac{i}{\omega\mu_o}\int_o^\infty {}^*\sqrt{s^2 - k^2}\, A(s) \cos xs\, ds + H_{x,\infty} = -\frac{i}{\omega\mu_o}\int_o^\infty \sqrt{s^2 - k_g^2}\, G(s) \cos xs\, ds \tag{43}$$

On using Eqs. (22) and (42) this condition becomes

$$-\frac{1}{\omega\mu_o}\int_o^\infty \left[\sqrt{s^2 - k_g^2} + {}^*\sqrt{s^2 - k^2}\right] A(s) \cos xs\, ds = \frac{khI}{2}\frac{H_1^{(1)}\left(k\sqrt{x^2 + h^2}\right)}{\sqrt{x^2 + h^2}}$$

$$= \frac{I}{\pi}\int_o^\infty F(h, s) \cos xs\, ds \tag{44}$$

where by the Fourier integral theorem the (dimensionless) function

$$F(h, s) = k\int_o^\infty \frac{h}{\sqrt{x^2 + h^2}} H_1^{(1)}\left(k\sqrt{x^2 + h^2}\right) \cos xs\, dx \tag{45}$$

It follows that

$$A(s) = -\frac{\mu_o \omega I}{\pi}\frac{F(h, s)}{\sqrt{s^2 - k_g^2} + {}^*\sqrt{s^2 - k^2}} \tag{46}$$

On consolidating the results, the fields above the earth are:

$$E_z(x,y) = -\frac{\mu_o \omega I}{\pi} \int_o^\infty \frac{e^{-y\,{}^*\sqrt{s^2-k^2}}\ F(h,s)\cos xs}{\sqrt{s^2-k_g^2} + {}^*\sqrt{s^2-k^2}}\, ds$$

$$-\frac{\mu_o \omega I}{4}\left[H_o^{(1)}(kr_-) - H_o^{(1)}(kr_+)\right] \qquad (47)$$

$$H_x(x,y) = -\frac{iI}{\pi} \int_o^\infty \frac{e^{-y\,{}^*\sqrt{s^2-k^2}}\ {}^*\sqrt{s^2-k^2}\ F(h,s)\cos xs}{\sqrt{s^2-k_g^2} + {}^*\sqrt{s^2-k^2}}\, ds$$

$$+\frac{ikI}{4}\left[\frac{h-y}{r_-} H_1^{(1)}(kr_-) + \frac{h+y}{r_+} H_1^{(1)}(kr_+)\right] \qquad (48)$$

$$H_y(x,y) = \frac{iI}{\pi} \int_o^\infty \frac{e^{-y\,{}^*\sqrt{s^2-k^2}}\ sF(h,s)\cos xs}{\sqrt{s^2-k_g^2} + {}^*\sqrt{s^2-k^2}}\, ds$$

$$+\frac{ikI}{4}\left[\frac{x}{r_-} H_1^{(1)}(kr_-) - \frac{x}{r_+} H_1^{(1)}(kr_+)\right] \qquad (49)$$

Below the surface of the earth the fields are:

$$E_z(x,y) = -\frac{\mu_o \omega I}{\pi} \int_o^\infty \frac{e^{y\sqrt{s^2-k_g^2}}\ F(h,s)\cos xs}{\sqrt{s^2-k_g^2} + {}^*\sqrt{s^2-k^2}}\, ds \qquad (50)$$

$$H_x(x,y) = \frac{iI}{\pi} \int_o^\infty \frac{e^{y\sqrt{s^2-k_g^2}}\ \sqrt{s^2-k_g^2}\ F(h,s)\cos xs}{\sqrt{s^2-k_g^2} + {}^*\sqrt{s^2-k^2}}\, ds \qquad (51)$$

$$H_y(x,y) = \frac{iI}{\pi} \int_0^{\infty} \frac{e^{y\sqrt{s^2 - k_g^2}} \, s\, F(h,s) \sin xs}{\sqrt{s^2 - k_g^2} + {}^*\sqrt{s^2 - k^2}} \, ds \tag{52}$$

On converting to the dimensionless parameters:

$$\begin{aligned} u &= s/k \\ x_1 &= kx \\ y_1 &= ky \\ h_1 &= kh \end{aligned} \tag{53}$$

Eq. (45) becomes

$$F(h_1, u) = \int_0^{\infty} \frac{h_1}{\sqrt{x_1^2 + h_1^2}} H_1^{(1)}\left(\sqrt{x_1^2 + h_1^2}\right) \cos ux_1 \, dx_1 \tag{54}$$

*and, for instance, with $y > 0$,*

$$E_z(x,y) = -\frac{\mu_o \omega I}{\pi} \int_0^{\infty} \frac{e^{-y_1 {}^*\sqrt{u^2 - 1}} \, F(h_1, u) \cos ux_1}{\sqrt{u^2 - k_g^2/k^2} + {}^*\sqrt{u^2 - 1}} \, du$$

$$- \frac{\mu_o \omega I}{4} \left[ H_o^{(1)}(k r_-) - H_o^{(1)}(k r_+) \right] \tag{55}$$

where

$$k_g^2/k^2 = \varepsilon/\varepsilon_o + i\sigma/\omega\varepsilon_o \quad . \tag{56}$$

## 6. PLAUSIBLE APPROXIMATIONS

Before proceeding with a numerical investigation of a particular case of the exact solution, it may be of interest to mention, without justification, some

plausible approximations. If the real part of the Hankel function $H_1^{(1)}$ is ignored, and its small-argument approximation

$$H_1^{(1)}(\upsilon) \cong -\frac{2i}{\pi\upsilon} \tag{57}$$

[see Eq. (10)] is used in the integral for F(h, u) even though the range of integration extends to infinity, then

$$F(h_1, u) \cong -\frac{2i}{\pi}\int_0^\infty \frac{h_1 \cos ux_1}{x_1^2 + h_1^2}\, dx_1 \tag{58}$$

This integral is a standard form, giving

$$F(h_1, u) \cong -ie^{-h_1 u} \tag{59}$$

Next if ${}^*\sqrt{u^2 - 1}$ is replaced by u, Eq. (55) gives

$$E_z(x, y) \cong \frac{i\mu_o \omega I}{\pi}\int_0^\infty \frac{e^{-(h_1 + y_1)u} \cos ux_1}{\sqrt{u^2 - k_g^2/k^2} + u}\, du - \frac{\mu_o \omega I}{4}\left[H_o^{(1)}(kr_-) - H_o^{(1)}(kr_+)\right] \tag{60}$$

Finally, if the Hankel functions $H_o^{(1)}$ are replaced by small argument approximations:

$$H_o^{(1)}(kr_\pm) \cong 1 - \frac{2i}{\pi}\, \ln \frac{2}{\gamma k r_\pm}$$

[see Eq. (10)], the result is

$$E_z(x, y) \cong \frac{i\mu_o \omega I}{2\pi}\left\{2\int_0^\infty \frac{e^{-(h_1 + y_1)u} \cos ux_1}{\sqrt{u^2 - k_g^2/k^2} + u}\, du + \ln \frac{\sqrt{x_1^2 + (y_1 + h_1)^2}}{\sqrt{x_1^2 + (y_1 - h_1)^2}}\right\} \tag{61}$$

This is similar to a form used by Wait[3] for highly conducting earths and low frequencies.†

## 7. NUMERICAL EXAMPLE

Using the exact solution [Eq. (55)], the real part of the electric field at the line current itself ($x = 0$; $y = h$) will be calculated numerically to within a few percent for the following case:

$$f = 3.5 \times 10^4 \text{ Hz}$$

$$\lambda = 8571.42857 \text{ m}$$

$$k = 7.33038287 \times 10^{-4} \text{ m}^{-1}$$

$$\sigma = 10^{-3} \text{ mho/m}$$

$$\varepsilon/\varepsilon_o = 10$$

$$\sigma/\omega\varepsilon_o = 514.2857143$$

$$h = 10 \text{ m}$$

$$y_1 = h_1 = 7.33038287 \times 10^{-3}$$

$$kr_+ = 2kh = 0.014660764$$

$$kr_- = 0$$

Then

$$\text{Re}\left[H_o^{(1)}(kr_-) - H_o^{(1)}(kr_+)\right] = J_o(0) - J_o(0.014660764)$$

$$= 5.38 \times 10^{-5}$$

by interpolation of tables,[4] and

$$\text{Re}[E_z] = -\frac{\mu_o \omega I}{\pi} \text{Re} \int_o^\infty (Q_r + iQ_i) F(h_1, u)\, du - \frac{\mu_o \omega I}{4} (5.38 \times 10^{-5}) \tag{62}$$

---

†Wait's formulation evidently contains a typographical error in which his symbols x and y were interchanged in three places.


3. Wait, J.R. (1961) On the impedance of a long wire suspended over the ground, Proc. IRE, 49, October, p 1576.

4. Cambi, E. (1948) Eleven and Fifteen-Place Tables of Bessel Functions of the First Kind to all Significant Orders, Dover, New York.

where

$$Q = Q_r + iQ_i = \frac{e^{-y_1 {}^*\sqrt{u^2 - 1}}}{\sqrt{u^2 - \varepsilon/\varepsilon_o - i\sigma/\omega\varepsilon_o} + {}^*\sqrt{u^2 - 1}} \tag{63}$$

The functions $Q_r$ and $Q_i$ in the interval $0 < u < 400$ are illustrated in Figure 3. Then if $F_r$ and $F_i$ are, respectively, the real and imaginary parts of $F(h_1, u)$,

$$Re\{E_z\} = -\frac{\mu_o \omega I}{8}\left[\frac{8}{\pi}\int_o^\infty Q_r F_r \, du - \frac{8}{\pi}\int_o^\infty Q_i F_i \, du + 1.08 \times 10^{-4}\right] \tag{64}$$

Now

$$F_r = h_1 \int_o^\infty \frac{J_1(\rho)}{\rho} \cos ux_1 \, dx_1 \tag{65}$$

with

$$\rho = \sqrt{x_1^2 + h_1^2} \tag{66}$$

Figure 4 shows a graph of $J_1(\rho)/\rho$ for $0 < x_1 < 20$. For large values of $x_1$, $J_1(\rho)/\rho$ approaches $\sqrt{2/\pi}\ \cos\ (x_1 - 3\pi/4)/x_1^{3/2}$ so that

$$\left| \int_{20}^\infty \frac{J_1(\rho)}{\rho} \cos ux_1 \, dx_1 \right| < \sqrt{2/\pi}\int_{20}^\infty \frac{dx}{x_1^{3/2}} = \sqrt{\frac{2}{\pi}}\,\frac{2}{\sqrt{20}} = 0.36$$

for all values of u. The portion of the integral from 0 to 20 is not larger than about 0.5, so it is concluded that $F_r$ is less than something of the order of $h_1$ or $7 \times 10^{-3}$ for all values of u, and will be much less than $h_1$ at large values of u. Inspection of Figure 3 shows that the area under the curve of $Q_r(u)$ is about 1, so that the first term inside the large bracket in Eq. (64) is not larger than about $7 \times 10^{-3}$.

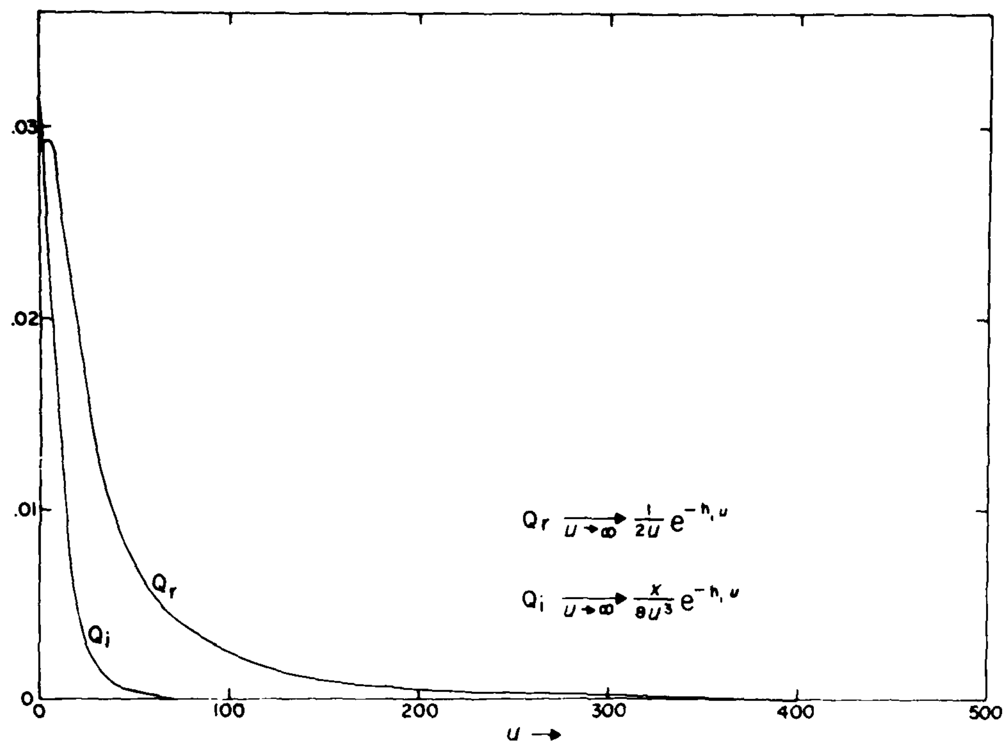


Figure 3. Graphs of the Functions $Q_r$ and $Q_i$

Now,

$$F_i = h_1 \int_0^\infty \frac{Y_1(\rho)}{\rho} \cos ux_1 \, dx_1 \tag{67}$$

may be written in terms of the small-argument approximation mentioned in Section 6, plus a remainder D which accounts for the difference between the exact and approximated function. This gives the exact equation

$$F_i = -\frac{2h_1}{\pi} \int_0^\infty \frac{1}{\rho^2} \cos ux_1 \, dx_1 + D(u) \tag{68}$$

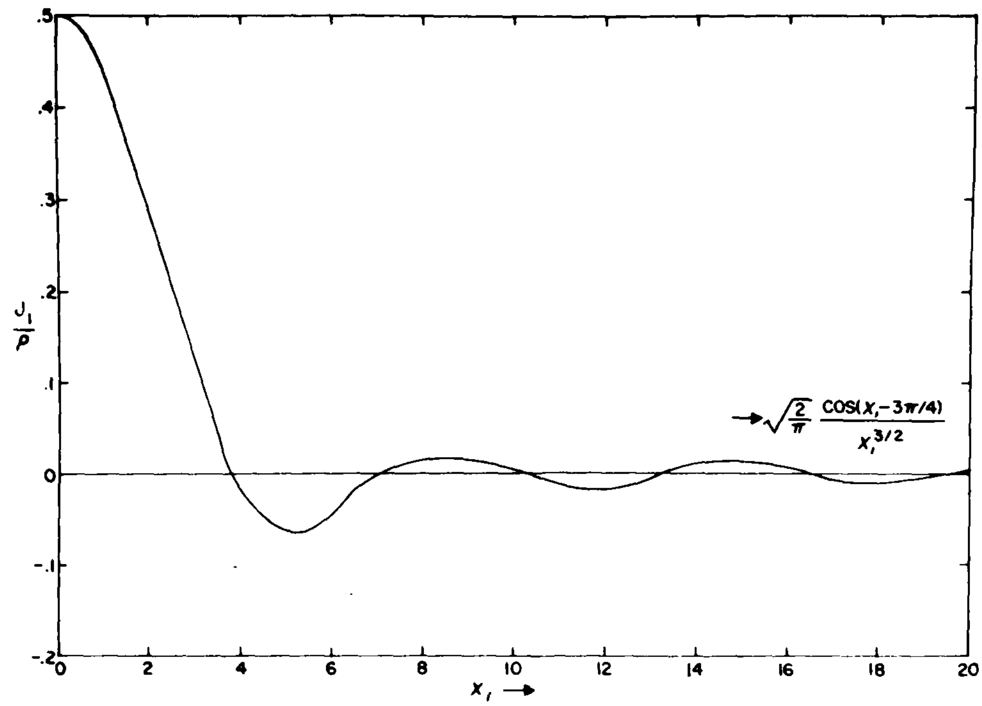


Figure 4. Graph of the Function $J_1(\rho)/\rho$

where

$$D(u) = \frac{2h_1}{\pi}\int_0^\infty \Delta \cos ux_1\, dx_1 \tag{69}$$

$$\Delta = \frac{\pi}{2\rho} Y_1(\rho) + \frac{1}{\rho^2} \tag{70}$$

Evaluating the integral of Eq. (68) as in Section 6,

$$F_i = -e^{-h_1 u} + D(u) \tag{71}$$

A graph of a portion of the function $\Delta$ is shown in Figure 5. Inspection shows that the portion of $\Delta$ for $x_1 < 10$ contributes a magnitude less than about $0.3(2h_1/\pi)$

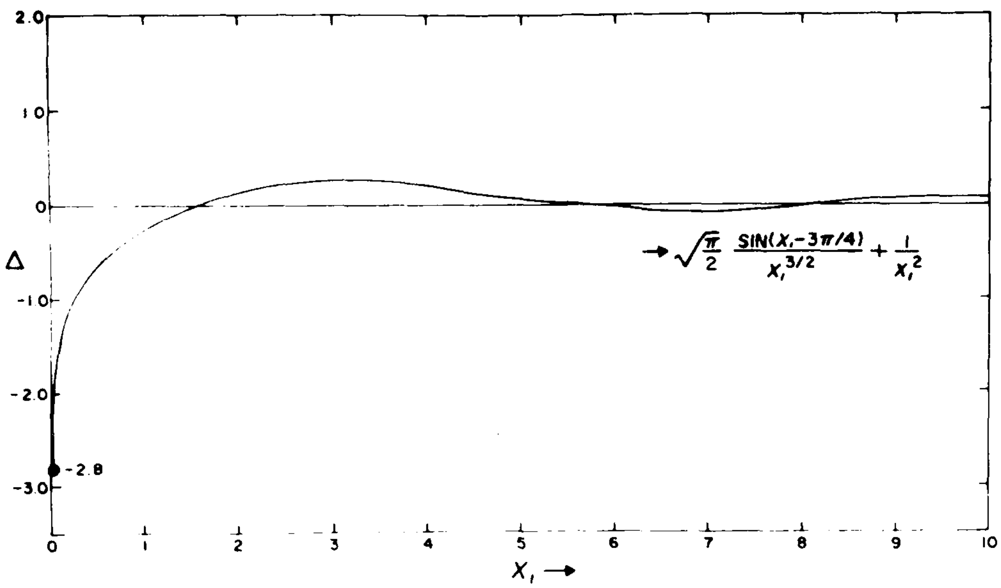


Figure 5. A Portion of the Function $\Delta(x_1)$

to the function D, for all values of u. For values of $x_1$ greater than 10, the limiting form

$$\Delta(x_1) \rightarrow \sqrt{\frac{\pi}{2}} \frac{\sin(x_1 - 3\pi/4)}{x_1^{3/2}} + \frac{1}{x_1^2} \tag{72}$$

represents Δ with an error less than about $10^{-3}$. The contribution to D from $x_1 > 10$ is then less than about

$$\frac{2h_1}{\pi} \int_{10}^{\infty} \left( \sqrt{\frac{\pi}{2}} \frac{1}{x_1^{3/2}} + \frac{1}{x_1^2} \right) dx \cong 0.9 \left( \frac{2h_1}{\pi} \right) \tag{73}$$

for all values of u. Thus for $0 \leq u \leq 100$, $D(u) \leq 2h_1/\pi = 0.005$, or less than 1 percent of $e^{-h_1 u}$ which is between 1 and 0.480 in this range. At larger values of u, the entire factor $F_i$ becomes quite unimportant because the multiplying factor $Q_i(u)$ in Eq. (64) is then much less than 1 percent of its initial value, as shown in Figure 3.

The product function $-Q_i F_i$ in Eq. (64) is shown in Figure 6 for the important part of its range. A numerical integration gives

$$\int_0^{\infty} -Q_i(u)\, F_i(u)\, du \;\cong\; 0.33 \tag{74}$$

and hence

$$\mathrm{Re}[E_z] = -\frac{\mu_o \omega I}{8}\left[\frac{8}{\pi}(0.33) - 0(\text{[illegible]}\ 10^{-3}) + 1.1 \times 10^{-4}\right]$$

$$= -0.84 \frac{\mu_o \omega I}{8} \tag{75}$$

The corresponding resistivity of the line is $0.84\, \mu_o \omega / 8 = 0.029\ \Omega/\mathrm{m}$.†

---

†Based on the previously mentioned approximate formulation (Section 6), Wait[1,5] gave two terms of a series for the resistivity:

$$\frac{\mu_o \omega}{8}\left[1 - \frac{8\sqrt{2}}{3\pi}\sqrt{\sigma \mu_o \omega}\; h + \ldots\right] .$$

For the above numerical example, the quantity $(8\sqrt{2}/3\pi)\sqrt{\sigma\mu_o\omega}\, h = 0.200$, and the corresponding line resistivity is $0.80\, \mu_o \omega / 8\ \Omega/\mathrm{m}$.

5. Wait, J.R. (1974) Comments on 'The horizontal wire antenna over a conducting or dielectric half space: current and admittance,' Radio Science 9:1165.

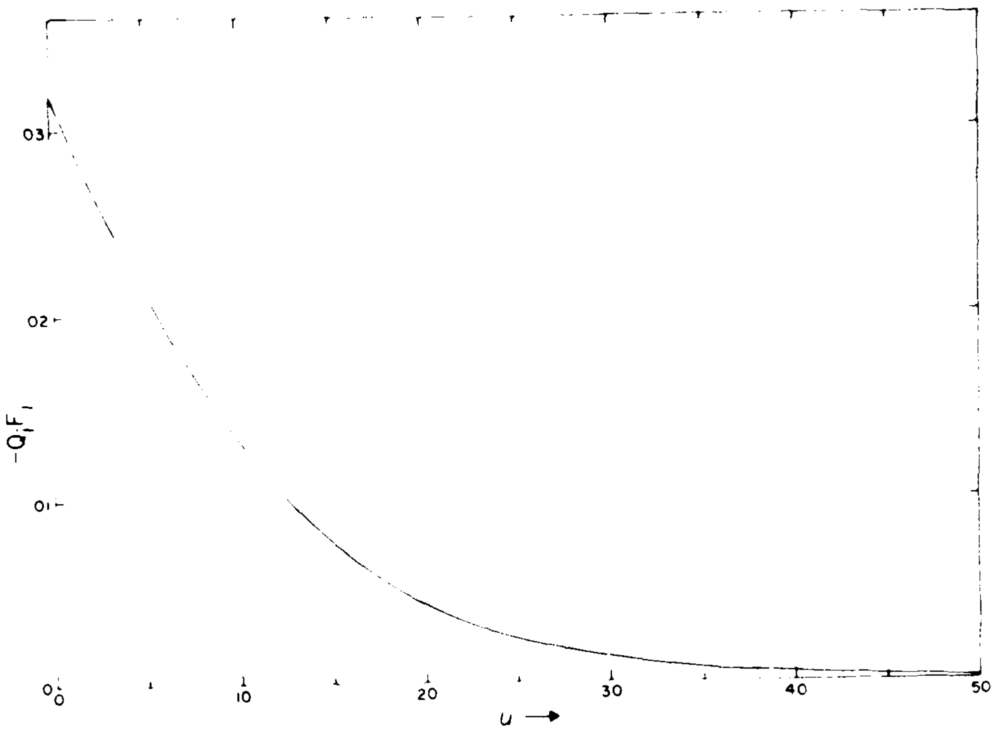


Figure 6. Graph of $-Q_i F_i$

## References


1. Carson, J.R. (1926) Wave propagation in overhead wires with ground return, Bell System Technical Journal 5:539-554.

2. Abramowitz, A., and Stegun, I.A. (1964) Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series 55, U.S. Government Printing Office, Washington, D.C.

3. Wait, J.R. (1961) On the impedance of a long wire suspended over the ground, Proc. IRE, 49, October, p 1576.

4. Cambi, E. (1948) Eleven and Fifteen-Place Tables of Bessel Functions of the First Kind to all Significant Orders, Dover, New York.

5. Wait, J.R. (1974) Comments on 'The horizontal wire antenna over a conducting or dielectric half space: current and admittance,' Radio Science 9:1165.




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